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On the ruin probability for physical fractional Brownian motion

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Abstract

We derive the exact asymptotic behavior of the ruin probability $P\{X(t) > x \text{ for some } t > 0\}$ for the process $X(t) = \int_0^t \xi(s) ds - ct$, with respect to level x which tends to infinity. We assume that the underlying process $\xi(t)$ is a.s. continuous stationary Gaussian with mean zero and correlation function regularly varying at infinity with index $-a \in (-1, 0)$.

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1. Introduction

Let $\xi(t)$, $t \geq 0$, be a stationary, a.s. continuous Gaussian random process with mean zero, variance one and covariance function $r(t)$. We assume that $r(t)$ is regularly varying at infinity with index $-a$, $0 < a < 1$, and write $r(t) \in \text{RV}_{-a}$. It means that for any positive t ,

$$\lim_{s \rightarrow \infty} \frac{r(ts)}{r(s)} = t^{-a}. \quad (1)$$

The subject of interest is the following Gaussian process with stationary increments which we like to call *physical fractional Brownian motion* (PFBM) with linear

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drift,

$$X(t) = \int_0^t \xi(s) ds - ct, \quad (2)$$

for a positive c . This is an integrated fractional Brownian motion, hence a Gaussian process with a.s. continuous first derivative. This model occurs in the ruin problem in finances, in storage theory, in information traffic theory, etc. The ruin probability with respect to the level x , is

$$\mathcal{P}(x) = P(\exists t \geq 0 : X(t) > x). \quad (3)$$

We evaluate the exact asymptotic behavior of the ruin probability as x tends to infinity. Note that if $a > 1$ then $r(t)$ is summable; this case was considered in [2], by introducing a generalized Pickands' constant. Our considerations are mainly based on properties of regularly varying functions and on application of traditional Gaussian double-sum techniques. For this matter, consider the rough behavior of variance and variance of increments for the process $X(t)$. We have,

$$\begin{aligned} \text{Var}(X(v+t) - X(v)) &= \text{Var} X(t) = 2 \int_0^t (t-s)r(s) ds \\ &= 2t^2 r(t) \int_0^1 (1-s) \frac{r(st)}{r(t)} ds. \end{aligned} \quad (4)$$

By (1) and Karamata's direct theorem [1, Proposition 1.5.8], we immediately get

$$\begin{aligned} \text{Var} X(t) &= 2t \int_0^t r(s) ds - 2 \int_0^t sr(s) ds \sim \frac{2t^2 r(t)}{1-a} - \frac{2t^2 r(t)}{2-a} \\ &= \frac{2t^2 r(t)}{(1-a)(2-a)} \end{aligned} \quad (5)$$

as $t \rightarrow \infty$, where $f(x) \sim g(x)$ denotes equivalence at $\omega = \infty$ (in this case), i.e. $f(x)/g(x) \rightarrow 1$ as $x \rightarrow \omega$.

The simplest case of regularly varying function is $r(t) \sim t^{-a}$ as $t \rightarrow \infty$, then $\text{Var}(X(t) - X(s)) \sim \text{const}(t-s)^{2-a}$, that is the variance of increments behaves at infinity like a fractional Brownian motion with the Hurst parameter $H = 1 - a/2$. On the other hand, the process $X(t)$ is a.s. differentiable, i.e. it exists *physically*. Note that the corresponding case analyzing the ruin probability for fractional Brownian motion with drift, was considered in [4].

To evaluate the asymptotic behavior of the ruin probability (3) we transform the probability in the following way. For $x > 0$

$$\begin{aligned} \mathcal{P}(x) &= P\left(\exists t > 0 : \int_0^t \xi(s) ds > x + ct\right) \\ &= P\left(\exists t > 0 : \int_0^{xt} \xi(s) ds > x + cxt\right) \end{aligned}$$

$$\begin{aligned} &= \mathbb{P}\left(\exists t > 0 : \frac{1}{1+ct} \int_0^{xt} \xi(s) \, ds > x\right) \\ &= \mathbb{P}\left(\sup_{t>0} \frac{1}{1+ct} \int_0^{xt} \xi(s) \, ds > x\right). \end{aligned}$$

Similarly, to (4) we derive

$$\text{Var} \int_0^{xt} \xi(s) \, ds = 2x^2 r(x) \int_0^t (t-s) \frac{r(xs)}{r(x)} \, ds.$$

Since r is regularly varying then there exists $x_0 > 1$ such that $|r(x)| > 0$ for all $x > x_0$. In fact, it follows by (5) that $r(x) > 0$ for all $x > x_0$ since the variance of $X(t)$ is positive and tends to ∞ using (5) and the regular variation of $r(t)$. Introduce the family of Gaussian processes,

$$Z_x(t) = \frac{1}{x\sqrt{r(x)(1+ct)}} \int_0^{xt} \xi(s) \, ds, \quad t \geq 0, \quad x \geq x_0. \quad (6)$$

We have $\mathbb{E} Z_x(t) \equiv 0$,

$$\sigma^2(t, x) := \text{Var} Z_x(t) = \frac{2}{(1+ct)^2} \int_0^t (t-s) \frac{r(xs)}{r(x)} \, ds$$

and

$$\mathcal{P}(x) = \mathbb{P}\left(\max_{t>0} Z_x(t) > \frac{1}{\sqrt{r(x)}}\right).$$

To formulate the main result of the paper we first denote

$$\sigma_x^2 := \sup_{t \geq 0} \sigma^2(t, x).$$

Then, let $g(x)$ be a minimal root of the equation

$$g^2 r(gx) = r(x)^2, \quad (7)$$

i.e., with $R(x) := x^2 r(x)$, $g(x) = x^{-1} R^{\leftarrow}(x^2 r(x)^2)$. From here, using Proposition 1.5.7 and Theorem 1.5.12 [1], we get that $g(x) \in \text{RV}_{-a/(2-a)}$ as $x \rightarrow \infty$. In particular, from here it follows that $0 < g < 1$ for all sufficiently large x .

For example, if $r(x) \sim Cx^{-a}$ as $x \rightarrow \infty$ we have $g(x) \sim C^{1/(2-a)} x^{-a/(2-a)}$, and if $r(x) \sim C(\log x)^A x^{-a}$ with A some constant, then $g(x) \sim C^*(\log x)^{A/(2-a)} x^{-a/(2-a)}$, where $C^* = C^*(C, a, A) > 0$.

We denote by $\Phi(u)$ the distribution function of the unit normal law and by $\Psi(u) = 1 - \Phi(u)$ its tail. In addition let H_α denotes Pickands constant with parameter α , for a definition see e.g. [5, 6, 7]. Note that $0 < H_\alpha < \infty$. Then we derive the following asymptotic statement for the ruin probability.

Theorem 1. *Let $X(t)$ be the physical fractional Brownian motion with drift defined by (2). Let (1) be fulfilled. Then for the ruin probability and for g defined by (7)*

we have,

$$\mathcal{P}(x) \sim (\sigma^4(1-a)(2-a)4/a^2)^{-1/(2-a)} \frac{\sqrt{\pi}H_{2-a}\sigma\sqrt{r(x)}}{\sqrt{B}g(x)} \Psi\left(\frac{1}{\sqrt{r(x)}\sigma_x}\right)$$

as $x \rightarrow \infty$, where

$$\sigma^2 = \lim_{x \rightarrow \infty} \sigma_x^2 = \frac{(2-a)^{1-a}a^a}{2c^{2-a}(1-a)}, \quad B = \frac{c^2a^3}{8(2-a)}.$$

2. Proofs

2.1. Properties of $Z_x(t)$

By (5), the variance of $Z_x(t)$ can be approximated by

$$\sigma^2(t, x) \sim \frac{2(tx)^2r(tx)}{(1-a)(2-a)(1+ct)^2x^2r(x)} \rightarrow \frac{2t^{2-a}}{(1-a)(2-a)(1+ct)^2} =: \sigma^2(t) \quad (8)$$

as $x \rightarrow \infty$. The limit $\sigma^2(t)$ attains its maximum at the unique point $t_0 = (2-a)/ca$ and equals

$$\sigma^2 := \frac{(2-a)^{1-a}a^a}{2c^{2-a}(1-a)}.$$

Since $\sigma^2(t, x) \rightarrow 0$ as $t \rightarrow \infty$, for all x , we may denote

$$t_0(x) = \arg \max_{t \geq 0} \sigma^2(t, x) \quad \text{so that} \quad \sigma_x^2 = \sigma^2(t_0(x), x) = \max_{t \geq 0} \sigma^2(t, x).$$

In case $\sigma^2(t, x)$ attains its maximum in several points we mean that $t_0(x)$ can be *any* point at which σ_x^2 is attained. We are going to prove that the maximal value of $\sigma^2(t, x)$ tends to the maximal value σ^2 of its limit, as x tends to infinity, as well as $t_0(x)$ is unique for large x , tending to t_0 , the point of maximum of $\sigma^2(t)$.

Lemma 2. *Let the Gaussian centered process be given by (6). Then for $t_0(x)$ at which the maximum of $\sigma^2(t, x)$ is attained, is unique for large x and*

1. $\lim_{x \rightarrow \infty} \sigma^2(t_0(x), x) = \sigma^2$,
2. $\lim_{x \rightarrow \infty} t_0(x) = t_0$.

Proof. Suppose first that $t_0(x) \rightarrow t_0$ as $x \rightarrow \infty$, and note that by Theorem 1.5.2 [1], the convergence (8) holds locally uniform in t . Hence the Assertion 1 follows. Now assume that there exists a sequence x_n , $x_n \rightarrow \infty$ and such that $t_0(x_n) \rightarrow t_1 < \infty$ with $t_1 \neq t_0$. Then, by the same theorem, $\sigma^2(t_0(x_n), x_n) \rightarrow \sigma^2(t_1) < \sigma^2$, which contradicts the choice of $t_0(x)$. In the other case $t_0(x_n) \rightarrow \infty$, we have by a Potter bound [1, Theorem 1.5.6] that for positive constants $b < a$ and C_b , T , and for all $t \geq T$, $\text{Var } Z_x(t) \leq C_b t^{-b}$, which again contradicts the choice of $t_0(x)$. Thus the first statement follows.

Now note that $\sigma^2(t, x) = C\sigma^2(t)F(t, x)$ with

$$F(t, x) = t^{a-2} \int_0^t (t-s) \frac{r(xs)}{r(x)} ds$$

and C some positive constant. We have $\sigma^2(t) = \sigma^2 - (C_1 + o(1))(t - t_0)^2$ and by straightforward calculations that

$$\lim_{x \rightarrow \infty} F(t, x) = 1/(1-a)(2-a),$$

$$\lim_{x \rightarrow \infty} F'_t(t, x) = 0 = \lim_{x \rightarrow \infty} F''_{tt}(t, x)$$

uniformly in a neighborhood of t_0 . Hence $(\sigma^2(t, x))''_{tt}$, the second derivative of $\sigma^2(t, x)$, tends to $(\sigma^2(t))''_{tt}$, the second derivative of $\sigma^2(t)$ as $x \rightarrow \infty$ which is negative. This implies that $\sigma^2(t, x)$ is concave in a neighborhood of t_0 . Thus $\sigma^2(t, x)$ can have only one maximum, hence $t_0(x)$ is unique. \square

Now we consider the variance of increments of the process $Z_x(t)$. We have for $s < t$,

$$\begin{aligned} & E(Z_x(t) - Z_x(s))^2 \\ &= E \left(\frac{1}{1+ct} \frac{1}{x\sqrt{r(x)}} \int_0^{xt} \xi(v) dv - \frac{1}{1+cs} \frac{1}{x\sqrt{r(x)}} \int_0^{xs} \xi(v) dv \right)^2 \\ &= E \left(\left(\frac{1}{1+ct} - \frac{1}{1+cs} \right) \frac{1}{x\sqrt{r(x)}} \int_0^{xt} \xi(v) dv \right. \\ &\quad \left. + \frac{1}{1+cs} \frac{1}{x\sqrt{r(x)}} \int_{xs}^{xt} \xi(v) dv \right)^2 \\ &= \left(\frac{c(t-s)}{(1+ct)(1+cs)} \right)^2 \sigma_0^2(t, x) \\ &\quad - \frac{2c(t-s)}{(1+ct)(1+cs)} \frac{1}{x^2 r(x)} E \int_0^{xt} \xi_v dv \int_{xs}^{xt} \xi(v) dv \\ &\quad + \left(\frac{1}{1+cs} \right)^2 \sigma_0^2(t-s, x), \end{aligned} \tag{9}$$

where

$$\sigma_0^2(t, x) := 2 \int_0^t (t-u) \frac{r(xu)}{r(x)} du \tag{10}$$

is the variance $\sigma^2(t, x)$ of the process $Z_x(t)$ for $c = 0$.

First investigate the behavior of $\sigma_0^2(t, x)$.

Lemma 3. Let g be defined by (7). The following is valid:

1. For any $\varepsilon, T > 0$,

$$\lim_{x \rightarrow \infty} \frac{\sigma_0^2(g(x)t, x)}{r(x)} = \frac{2t^{2-a}}{(1-a)(2-a)}$$

uniformly in $t \in [\varepsilon, T]$.

2. For any a' , $1 > a' > a$, there exists x_0 and $L^* > 0$ such that for all $x \geq x_0$ and $t \geq 1$,

$$\frac{\sigma_0^2(g(x)t, x)}{r(x)} \geq L^* t^{2-a'}.$$

3. For any a' , $1 > a' > a$ and any $T > 0$ there exist positive constants L and x_0 such that for all $x > x_0$ and $t \in (0, T]$,

$$\sigma_0^2(t, x) \leq Lt^{2-a'}.$$

Proof. Define

$$\Sigma(xg(x)t) := x^2 r(x) \sigma_0^2(g(x)t, x) = 2xg(x)t \int_0^{xg(x)t} r(s) ds - 2 \int_0^{xg(x)t} sr(s) ds. \quad (11)$$

By (5) we have

$$\Sigma(x) \sim \frac{2x^2 r(x)}{(1-a)(2-a)} \in \text{RV}_{2-a}$$

as $x \rightarrow \infty$, and using the definition of g ,

$$\begin{aligned} \frac{\sigma_0^2(g(x)t, x)}{r(x)} &= \frac{\Sigma(xg(x)t)}{\Sigma(xg(x))} \frac{\Sigma(xg(x))}{x^2 r(x)^2} t^{2-a} \frac{2(xg(x))^2 r(xg(x))}{(1-a)(2-a)x^2 r(x)^2} \\ &= \frac{2t^{2-a}}{(1-a)(2-a)} \end{aligned}$$

as $x \rightarrow \infty$ and the convergence is uniform by Theorem 1.5.2 [1]. This gives the first assertion of the lemma.

In order to prove the second statement we change variables: $u = g(x)ts$ in the definition of $\sigma_0^2(t, x)$, and get

$$\begin{aligned} \frac{\sigma_0^2(g(x)t, x)}{r(x)} &= \frac{2}{r(x)} g^2(x) t^2 \int_0^1 (1-s) \frac{r(xg(x)ts)}{r(x)} ds \\ &= \frac{2g^2(x)r(xg(x))}{r^2(x)} t^2 \int_0^1 (1-s) \frac{r(xg(x)ts)}{r(xg(x))} ds. \end{aligned} \quad (12)$$

Now we use an inverse Potter's bound [1, Theorem 1.5.6] to conclude the proof of the second statement.

To prove the last claim, we directly use a Potter's bound [1, Theorem 1.5.6] for the fraction under the integral in (10), with appropriate choice of δ in the Theorem 1.5.6. The assertion follows by integrating. \square

Now we turn to (9) to estimate the variances of increments of the processes $Z_x(t)$.

Lemma 4. *Let the Gaussian centered process $Z_x(t)$ be given by (6). Then*

1. *For any $t, \epsilon, T > 0$,*

$$\lim_{x \rightarrow \infty} \frac{(1+ct)^2 E(Z_x(t) - Z_x(t + g(x)(t-s)))^2}{r(x)} = \frac{2|t-s|^{2-a}}{(1-a)(2-a)}$$

uniformly in s such that $\epsilon \leq |t-s| \leq T$.

2. *Let a' be such that $1 > a' > a$. There exist $\delta > 0$ and x_0 such that for all $x \geq x_0$, and t, s with $t-s \in [1, \delta/g(x)]$,*

$$\frac{(1+ct)^2 E(Z_x(t) - Z_x(t + g(x)(t-s)))^2}{r(x)} \geq L^* |t-s|^{2-a'}.$$

3. *Let a' be such that $1 > a' > a$. There exist L and x_0 such that for all $x \geq x_0$ and all t, s with $|t-s| \leq 1$ and $s, t \leq T$,*

$$E(Z_x(t) - Z_x(s))^2 \leq \frac{L|t-s|^{2-a'}}{(1+cs)^2}.$$

Proof. First, we bound the first two terms in the right-hand part of (9). From Lemma 3, we have for some constants L and L_1 ,

$$\left(\frac{c(t-s)}{(1+ct)(1+cs)} \right)^2 \sigma_0^2(t, x) \leq \frac{L(t-s)^2 t^{2-a'}}{(1+ct)^2 (1+cs)^2}. \quad (13)$$

Further, by Cauchy inequality

$$\begin{aligned} & \frac{2c(t-s)}{(1+ct)(1+cs)} \frac{1}{x^2 r(x)} E \int_0^{xt} \xi(v) dv \int_{xs}^{xt} \xi(v) dv \\ & \leq \frac{2c(t-s) \sigma_0(t, x) \sigma_0(t-s, x)}{(1+ct)(1+cs)}. \end{aligned} \quad (14)$$

Now the third statement follows from the inequalities of (13) and (14) and the third assertion of Lemma 3. To prove the first two statements we put $s_1 = t + g(x)(t-s)$ and apply the first two assertions of Lemma 3 for $t-s_1$. From (13) and (14) it follows that for such $t-s_1$ the first two terms in (9) are infinitely smaller than the third one. So the statements obviously follow from the first two assertions of Lemma 3. \square

Now we bound the probability

$$\mathcal{P}_T(x) = P \left(\max_{t \geq T} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right),$$

for large enough T because we want to restrict our considerations on a finite interval $[0, T]$. Denote by $[x]$ the integer part of x .

Lemma 5. *There exists a constant D such that for all T and x with Tx large,*

$$\mathbb{P}\left(\max_{t \geq T} Z_x(t) > \frac{1}{\sqrt{r(x)}}\right) \leq D \exp\left(-\frac{1}{Dr([xT])}\right).$$

Proof. By the definition of Z_x ,

$$\begin{aligned} \mathcal{P}_T(x) &= \mathbb{P}\left(\max_{t \geq T} \frac{1}{1+ct} \int_0^{xt} \xi(s) ds > x\right) \\ &= \mathbb{P}\left(\max_{t \geq xT} \int_0^t \xi(s) ds > x+ct\right) \\ &\leq \sum_{k=[xT]}^{\infty} \mathbb{P}\left(\max_{k \leq t \leq k+1} \int_0^t \xi(s) ds > x+ck\right). \end{aligned}$$

To apply Fernique's inequality, we note that the variance of $\int_0^t \xi(s) ds$ does not exceed $Ct^2r(t)$ for some C and for any large enough x and T by (5). One may take $C = (2 + \epsilon)/((1-a)(2-a))$, with any $\epsilon > 0$. Since $r(t)$ is regularly varying, we can change this bound on $[k, k+1]$ by $Ck^2r(k)$, where C is the same constant for all $k \geq [xT]$. Now, use Fernique inequality to get

$$\begin{aligned} \mathcal{P}_T(x) &\leq \sum_{k=[xT]}^{\infty} C_1 \exp\left(-\frac{1}{2} \frac{(x+ck)^2}{Ck^2r(k)}\right) \\ &\leq \sum_{k=[xT]}^{\infty} C_1 \exp\left(-\frac{c^2}{2Cr(k)}\right) \leq C_2 \exp\left(-\frac{c^2}{2Cr([xT])}\right) \\ &\leq D \exp\left(-\frac{1}{2Dr([xT])}\right), \end{aligned}$$

for some $C_2 > C_1$ and $D = \max(C_2, C/c^2)$, since the series converges faster than geometric progression. \square

In virtue of Lemma 5 we deal now with the process $Z_x(t)$ on $[0, T]$, where we select $T > t_0$ later. We need to analyze the behavior of the variance $\sigma^2(t, x)$ of the process $Z_x(t)$ near the point $t_0 = (2-a)/(ca)$, that is, the point of maximum of the limit variance $\sigma^2(t)$. See (8) for definitions.

Lemma 6. *For the variance $\sigma^2(t, x)$ of the process $Z_x(t)$,*

$$\lim_{x \rightarrow \infty} \frac{d}{dt} \sigma^2(t, x) = \frac{d}{dt} \sigma^2(t)$$

and

$$\lim_{x \rightarrow \infty} \frac{d^2}{dt^2} \sigma^2(t, x) = \frac{d^2}{dt^2} \sigma^2(t)$$

uniformly in a neighborhood of t_0 .

Proof. We use the following expression for $\sigma^2(t, x)$,

$$\sigma^2(t, x) = \frac{2}{(1 + ct)^2 x^2 r(x)} \int_0^{xt} \int_0^v r(s) ds dv, \quad (15)$$

hence the variance is twice differentiable. The derivatives of the integral are

$$\frac{d}{dt} \int_0^{xt} \int_0^v r(s) ds dv = x \int_0^{xt} r(s) ds, \quad (16)$$

$$\frac{d^2}{dt^2} \int_0^{xt} \int_0^v r(s) ds dv = x^2 r(xt). \quad (17)$$

By Karamata's theorem, [3, p. 15],

$$\lim_{x \rightarrow \infty} \frac{\int_0^{xt} r(s) ds}{xtr(xt)} = \frac{1}{1 - a}$$

and

$$\lim_{x \rightarrow \infty} \frac{\int_0^{xt} \int_0^v r(s) ds dv}{(xt)^2 r(xt)} = \frac{1}{(2 - a)(1 - a)},$$

which was derived already directly. Both the above limit relations are uniform in any interval $t \in [\varepsilon, T]$, thus in any finite neighborhood of t_0 . The statements of the lemma follow now by straightforward calculations.

2.2. Maximum on a finite interval

We need a slight generalization of an approximation of the distribution of the maxima of Gaussian processes on a finite interval. We need $H_\alpha(T)$ Pickands' constants for the finite intervals $[0, T]$, being continuous in T , and use the relation that $H_\alpha(T)/T \rightarrow H_\alpha$ as $T \rightarrow \infty$, see [5]. Let $\{X_u(t), t \geq 0\}$, $u > 0$, be a family of a.s. continuous Gaussian processes with $EX_u(t) \equiv 0$, $EX_u(t)^2 \equiv 1$, for all u with correlation function $r_u(t, s) = EX_u(t)X_u(s)$. We suppose that there exists a positive normalization b_u , tending to zero as $u \rightarrow \infty$, such that

- D1. For some $\alpha > 0$ and any t, s : $\lim_{u \rightarrow \infty} u^2(1 - r_u(b_u t, b_u s)) = |t - s|^\alpha$.
 D2. For some $\alpha_1 > 0$ and C and for all t, s, u : $u^2(1 - r_u(b_u t, b_u s)) \leq C|t - s|^{\alpha_1}$.

Lemma 7. Assume that the above family of Gaussian processes $X_u(t)$ satisfies the conditions D1, D2. Then for any T ,

$$P\left(\max_{t \in [0, Tb_u]} X_u(t) > u\right) = H_\alpha(T) \Psi(u)(1 + o(1))$$

as $u \rightarrow \infty$.

Proof. We start with well-known evaluations, (compare with lines on p. 14 [7]).

$$\begin{aligned} & \mathbb{P}\left(\max_{t \in [0, Tb_u]} X_u(t) > u\right) \\ &= \frac{1}{\sqrt{2\pi u}} e^{-u^2/2} \int_{-\infty}^{\infty} e^{w-w^2/2u^2} \mathbb{P}\left(\max_{t \in [0, T]} \chi_u(t) > w \mid \chi_u(0) = 0\right) dw, \end{aligned}$$

where $\chi_u(t) = u(X_u(b_u t) - u) + w$. We then have,

$$\begin{aligned} \mathbb{E}(\chi_u(t) \mid \chi_u(0) = 0) &= -u^2(1 - r_u(b_u t, 0)) + w(1 - r_u(b_u t, 0)), \\ \text{Var}(\chi_u(t) - \chi_u(s) \mid \chi_u(0) = 0) &= 2u^2(1 - r_u(b_u t, b_u s)) - u^2(r_u(b_u t, 0) - r_u(b_u s, 0))^2. \end{aligned}$$

We estimate the second term in the right-hand part of the last relation by using the triangle inequality for the semi-norm $\|t, s\| = u\sqrt{2 - 2r_u(b_u t, b_u s)}$. We have,

$$\begin{aligned} (\|t, 0\|^2 - \|s, 0\|^2)^2 &\leq (\|t, 0\| - \|s, 0\|)^2 (\|t, 0\| + \|s, 0\|)^2 \\ &\leq \|t, s\|^2 (\|t, 0\| + \|s, 0\|)^2 = 2|t - s|^\alpha o(1) \end{aligned}$$

as $u \rightarrow \infty$. Moreover, it follows that:

$$(\|t, 0\|^2 - \|s, 0\|^2)^2 \leq C|t - s|^{\alpha_1}$$

for all t, s . The proof follows now word by word the lines of the corresponding proof in [7]. \square

Corollary 8. *Let assumption D2 be fulfilled, and assume*

D1a. *There exist a function W_u , positive numbers w_1 and w_2 with $w_1 \leq W_u \leq w_2$ for all u , and α such that for any t, s : $\lim_{u \rightarrow \infty} W_u^{-1} u^2(1 - r_u(b_u t, b_u s)) = |t - s|^\alpha$.*

Then for any T ,

$$\mathbb{P}\left(\max_{t \in [0, Tb_u]} X_u(t) > u\right) = H_\alpha(TW_u^{1/\alpha})\Psi(u)(1 + o(1))$$

as $u \rightarrow \infty$.

Proof. Suppose first that there exists $W = \lim_{u \rightarrow \infty} W_u$. Consider the family of Gaussian processes $X_u(W^{-1/\alpha}t)$ and apply Lemma 7. Then, using monotonicity of the probability with respect to T , we get the assertion of the corollary. Considering now all limit points of the bounded function W_u , we get the assertion in the general case. \square

We need a simple estimation for a tail of the two-dimensional Gaussian distribution when its correlation is close to 1. Let X, Y be a Gaussian vector with means zero, variances one and correlation r which can depend on the level u in the probability

$$P(u; r) = \mathbb{P}(X > u, Y > u).$$

Lemma 9. Suppose that $r = r_u$ is such that $\liminf_{u \rightarrow \infty} u^2(1 - r) \geq A > 0$, for some A . Then

$$\limsup_{u \rightarrow \infty} P(u; r) / \Psi(u) \leq e^{-A/4}.$$

Proof. Note that

$$P(X > u, Y > u) \leq P(X + Y > 2u)$$

and $E(X + Y)^2 = 2 + 2r$. Now using for u positive

$$P(X + Y > 2u) \leq \frac{\sqrt{2 + 2r}}{2u\sqrt{2\pi}} \exp\left(-\frac{4u^2}{2(2 + 2r)}\right)$$

and

$$-\frac{4u^2}{2(2 + 2r)} = -\frac{u^2}{2} - \frac{u^2(1 - r)}{2(1 + r)} \leq -\frac{u^2}{2} - \frac{u^2(1 - r)}{4},$$

the claim follows. \square

2.3. Proof of Theorem 1

The statement is proved by considering the probability of ruin on subintervals. In Lemma 5 it is shown, that the probability of ruin in $t \in (T, \infty)$ is asymptotically negligible with respect to the probability of ruin on $[0, T]$. We show that this probability is also negligible for $t \notin (t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$. Hence we concentrate on the subinterval around t_0 , first.

(A) *Subinterval around t_0 :* We start by dealing with the subinterval around t_0 . By Lemma 2, for all sufficiently large x , $t_0(x) \in [t_0 - \delta, t_0 + \delta]$ for some positive δ . Using Lemma 6, Taylor expansion for the function $\sigma(t)/\sigma$ at point t_0 , and again Lemma 2, we get that for sufficiently small ϵ one can choose δ such that for all $t \in [t_0 - \delta, t_0 + \delta]$ and all sufficiently large x ,

$$1 - (1 + \epsilon)B(t - t_0(x))^2 \leq \frac{\sigma(t, x)}{\sigma_x} \leq 1 - (1 - \epsilon)B(t - t_0(x))^2, \quad (18)$$

where $B = c^2 a^3 / (8(2 - a))$. Denote

$$t_k = t_0(x) + kTg(x) \quad \text{and} \quad \Delta_k = [t_k, t_{k+1}], \quad k \in \mathbb{Z}$$

and cover the interval $[t_0 - \delta, t_0 + \delta]$ with intervals $\Delta_k \subset [t_0 - \delta, t_0 + \delta]$ where $g(x)$ is defined in (7). Denote this union of such intervals Δ_k by J_δ . First, we consider the process $Z_x(t)$ on $t \in \Delta_k \subset J_\delta$. We have for $k \geq 0$

$$1 - (1 + \epsilon)B(t_{k+1} - t_0(x))^2 \leq \frac{\sigma(t, x)}{\sigma_x} \leq 1 - (1 - \epsilon)B(t_k - t_0(x))^2, \quad t \in \Delta_k$$

and $k < 0$

$$1 - (1 + \epsilon)B(t_k - t_0(x))^2 \leq \frac{\sigma(t, x)}{\sigma_x} \leq 1 - (1 - \epsilon)B(t_{k+1} - t_0(x))^2, \quad t \in \Delta_k.$$

For k such that $(t_k - t_0(x))^2 \leq \delta/B(1 + \epsilon)$ which means for k such that $\Delta_k \subset J_{\delta'}$ with $\delta' = \min(\delta, \sqrt{\delta/B(1 + \epsilon)})$, we get the upper and lower bounds

$$\begin{aligned} & \mathbb{P}\left(\max_{\Delta_k} Z_x(t) > \frac{1}{\sqrt{r(x)}}\right) \\ &= \mathbb{P}\left(\max_{\Delta_k} \frac{Z_x(t)\sigma(t, x)}{\sigma(t, x)\sigma_x} > \frac{1}{\sqrt{r(x)}\sigma_x}\right) \\ &\geq \mathbb{P}\left(\max_{\Delta_k} \frac{Z_x(t)}{\sigma(t, x)} > \frac{1}{(1 - (1 \pm \epsilon)B(\tilde{t}_k - t_0(x))^2)\sqrt{r(x)}\sigma_x}\right), \end{aligned} \quad (19)$$

where $\tilde{t}_k = t_k$ or t_{k+1} , depending on the approximation. Denote

$$u_k = \frac{1}{(1 - b(\tilde{t}_k - t_0(x))^2)\sqrt{r(x)}\sigma_x},$$

where $b = (1 \pm \epsilon)B$, depending on situation. To investigate the correlation function of the process $Z_x(t)/\sigma(t, x)$ on such an interval Δ_k introduce the Gaussian process $Y_x(t) = Z_x(t_k + t)/\sigma(t_k + t, x)$, $t > 0$. The approximating probability in (19) can be written as

$$\mathbb{P}\left(\max_{\Delta^*} Y_x(t) > u_k\right),$$

where $\Delta^* = [0, g(x)T]$. By Lemma 4 we have

$$\lim_{x \rightarrow \infty} \frac{(1 + ct_k)^2 \mathbb{E}(Z_x(t_k + g(x)t) - Z_x(t_k + g(x)s))^2}{r(x)} = \frac{2|t - s|^{2-a}}{(1-a)(2-a)}.$$

Since $\lim_{x \rightarrow \infty} \sigma(t_k + g(x)t)/\sigma(t_k) = 1$ uniformly in $t \in \Delta^*$, and $(\sigma(t_k + g(x)t, x) - \sigma(t_k + g(x)s, x))^2 = O(g(x)^2) = o(r(x))$ as $x \rightarrow \infty$, we get that

$$\lim_{x \rightarrow \infty} \frac{(1 + ct_k)^2 \sigma^2(t_k, x) \mathbb{E}(Y_x(g(x)t) - Y_x(g(x)s))^2}{r(x)} = \frac{2|t - s|^{2-a}}{(1-a)(2-a)}.$$

Thus for the covariance function $\rho(t, s)$ of $Y_x(t)$ we have,

$$\lim_{x \rightarrow \infty} W_k u_k^2 (1 - \rho(g(x)t, g(x)s)) = |t - s|^{2-a},$$

where

$$W_k = \sigma^2(t_k)(1 - a)(2 - a)\sigma_x^2(1 - b(t_k - t_0(x))^2)^2(1 + ct_k)^2.$$

(B) *Upper bound:* $W_k = W_k(x)$ satisfies conditions of Corollary 8 by the choice δ' , so by Corollary 8,

$$\mathbb{P}\left(\max_{\Delta_0} Y_x(t) > u_k\right) = H_{2-a}(W_k^{-1/(2-a)}T)\Psi(u_k)(1 + o(1)) \quad (20)$$

as $x \rightarrow \infty$ uniformly for k such that $\Delta_k \subset J_{\delta'}$ since u_k tends uniformly to infinity as $x \rightarrow \infty$. Hence

$$\mathbb{P}\left(\max_{t \in J_{\delta'}} Z_x(t) > \frac{1}{\sqrt{r(x)}}\right) \leq (1 + \gamma_1(x)) \sum_{k: \Delta_k \subset J_{\delta'}} H_{2-a}(W_k^{-1/(2-a)}T) \Psi(u_k), \quad (21)$$

where $\gamma_1(x) \rightarrow 0$ as $x \rightarrow \infty$. In the following we use further functions $\gamma_n(x)$, $n \geq 2$, with $\gamma_n(x) \rightarrow 0$ as $x \rightarrow \infty$. Using the mentioned property of Pickands' constant $H_a(T)$, we get for large T

$$\begin{aligned} & \sum_k H_{2-a}(W_k^{-1/(2-a)}T) \Psi(u_k) \\ &= \sum_k W_k^{-1/(2-a)}T \frac{H_{2-a}(W_k^{-1/(2-a)}T)}{W_k^{1/(2-a)}T} \Psi(u_k) \\ &= (1 + o_T(1))TH_{2-a} \sum_k W_k^{-1/(2-a)} \Psi(u_k) \\ &= (1 + o_T(1)) \frac{(1 + \gamma_2(x))TH_{2-a}}{\sqrt{2\pi}} \sum_k W_k^{-1/(2-a)} u_k^{-1} e^{-u_k^2/2}, \end{aligned} \quad (22)$$

where $o_T(1)$ denotes a term tending to 0 as $T \rightarrow \infty$. Recall that $u_k^{-1} = \sqrt{r(x)}\sigma_x(1 - bT^2k^2g^2(x))$. For u_k^2 we have

$$\begin{aligned} u_k^2 &= \frac{1}{r(x)\sigma_x^2} + \frac{1}{r(x)\sigma_x^2} \left(\frac{1}{(1 - bk^2T^2g^2(x))^2} - 1 \right) \\ &= \frac{1}{r(x)\sigma_x^2} + \frac{1}{r(x)\sigma_x^2} \frac{2bk^2T^2g^2(x) - b^2k^4T^4g^4(x)}{(1 - bk^2T^2g^2(x))^2} \\ &= \frac{1}{r(x)\sigma_x^2} + \frac{1}{r(x)\sigma_x^2} 2bk^2T^2g^2(x)(1 + o(\delta')) \\ &\geq \frac{1}{r(x)\sigma_x^2} + \frac{1}{r(x)\sigma_x^2} 2bk^2T^2g^2(x)(1 - \delta'), \end{aligned} \quad (23)$$

where depending on the approximation k is replaced by $k+1$. But this has no influence on the following asymptotic derivations. Thus denote by

$$x_k = \frac{\sqrt{b}Tg(x)}{\sqrt{r(x)}\sigma_x} k$$

and use that $g^2(x) = o(r(x))$, we have $x_k - x_{k-1} \rightarrow 0$ as $x \rightarrow \infty$. From the inequality

$$\frac{u_k^2}{2} \geq \frac{1}{2r(x)\sigma_x^2} + x_k^2(1 - \delta')$$

it follows that we can apply the dominated convergence theorem for the sum in right-hand part of (22). If $|k| = o(1/Tg(x))$ then $W_k = (1 + o(1))\sigma^4(1 - a)(2 - a)(1 +$

$ct_0)^2 = (1 + o(1))\sigma^4(1-a)(2-a)4/a^2$. With these approximations we get by breaking the sum on k into two parts: $|k| \leq r_0 := r^{1/4}(x)/Tg(x)$ and the remaining k 's:

$$\begin{aligned}
 & \sum_k W_k^{-1/(2-a)} u_k^{-1} e^{-u_k^2/2} \\
 &= W_0^{-1/(2-a)} (1 + O(\delta')) \sqrt{r(x)} \sigma_x \\
 & \quad \times \sum_{|k| \leq r_0} \exp\left(-\frac{1}{2r(x)\sigma_x^2} - x_k^2(1 + o(1))\right) \\
 & \quad + O\left(\sqrt{r(x)} \sum_{|k| > r_0} \exp\left(-\frac{1}{2r(x)\sigma_x^2} - x_k^2(1 - \delta')\right)\right) \\
 &= \frac{W_0^{-1/(2-a)} r(x) \sigma_x^2 (1 + O(\delta')) \exp\left(-\frac{1}{2r(x)\sigma_x^2}\right)}{\sqrt{b}Tg(x)} 2 \int_0^{a(x)} \exp(-z^2(1 + o(1))) dz \\
 & \quad + O\left(r(x)g^{-1}(x) \exp\left(-\frac{1}{2r(x)\sigma_x^2}\right) 2 \int_{a(x)}^{a^*(x)} \exp(-z^2(1 - \delta')) dz\right) \\
 &= (1 + o(1)) \frac{(\sigma^4(1-a)(2-a)4/a^2)^{-1/(2-a)} r(x) \sigma_x^2 (1 + O(\delta')) \sqrt{\pi}}{\sqrt{b}Tg(x)} \\
 & \quad \times \exp\left(-\frac{1}{2r(x)\sigma_x^2}\right), \tag{24}
 \end{aligned}$$

where $a(x) = \sqrt{b}r^{-1/4}(x)/\sigma_x \rightarrow \infty$ and $a^*(x) = \sqrt{b}\delta' r^{-1/2}(x)/\sigma_x \rightarrow \infty$ as $x \rightarrow \infty$.

Finally, combining the approximations

$$\begin{aligned}
 & P\left(\max_{t \in [t_0 - \delta', t_0 + \delta']} Z_x(t) > \frac{1}{\sqrt{r(x)}}\right) \leq (1 + \gamma_3(x))(1 + o_T(1)) \\
 & \quad \times \frac{\sqrt{\pi}H_{2-a}(\sigma^4(1-a)(2-a)4/a^2)^{-1/(2-a)}}{\sqrt{b}} \\
 & \quad \times \frac{\sigma_x \sqrt{r(x)}}{g(x)} \Psi\left(\frac{1}{\sqrt{r(x)}\sigma_x}\right). \tag{25}
 \end{aligned}$$

This approximation does not depend on the choice of \tilde{t}_k .

(C) *Lower bound:* To estimate the probability from below we first introduce the grid $\mathcal{R}_\kappa = \{ikg(x), i \in \mathbf{Z}\}$, $\kappa > 0$, and denote $\Delta'_\kappa = \Delta_\kappa \cap \mathcal{R}_\kappa$. We consider again the union $J_{\delta'}$.

We have

$$\begin{aligned}
 & \mathbb{P} \left(\max_{t \in [t_0 - \delta', t_0 + \delta']} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right) \\
 & \geq \mathbb{P} \left(\max_{t \in [t_0 - \delta', t_0 + \delta'] \cap \mathcal{R}_\kappa} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right) \\
 & \geq \sum_{k: A'_k \subset J_{\delta'}} \mathbb{P} \left(\max_{t \in A'_k} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right) \\
 & \quad - \sum_{k, l, k < l: A'_k, A'_l \subset J_{\delta'}} \mathbb{P} \left(\max_{t \in A'_k} Z_x(t) > \frac{1}{\sqrt{r(x)}}, \max_{t \in A'_l} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right). \quad (26)
 \end{aligned}$$

To estimate the first sum we use the following easy asymptotic relation which follows from the proof of Lemma 6 (weak convergence of χ_u is important). In fact, it is the relation (15.8) from [7].

Corollary 10. *Under the assumptions of Lemma 6, for any $\epsilon > 0$ one can find $\kappa > 0$ such that*

$$\lim_{u \rightarrow \infty} \mathbb{P} \left(\max_{t \in J_{\delta'}} X_u(t) > u, \max_{t \in J_{\delta'} \cap \mathcal{R}_\kappa} X_u(t) \leq u \right) / \Psi(u) \leq \epsilon. \quad (27)$$

From Corollary 10 and the above asymptotic estimations for the upper bound (meaning now $b = B(1 + \epsilon)$) we get that for any $\epsilon > 0$ one can find a small positive κ such that

$$\begin{aligned}
 & \sum_{k: A'_k \subset J_{\delta'}} \mathbb{P} \left(\max_{t \in A'_k} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right) \\
 & \geq (1 - \epsilon)(1 - \gamma_3(x)) \frac{\sqrt{\pi} H_{2-a}(\sigma^4(1-a)(2-a)4/a^2)^{-1/(2-a)} \sigma_x \sqrt{r(x)}}{\sqrt{b} g(x)} \\
 & \quad \times \Psi \left(\frac{1}{\sqrt{r(x)} \sigma_x} \right). \quad (28)
 \end{aligned}$$

Now, we consider the double sum in the right-hand side of (26). Denote by

$$A(x) = \frac{\sqrt{r(x)}}{g(x)} \Psi \left(\frac{1}{\sqrt{r(x)} \sigma_x} \right).$$

We are going to show that the order of the double sum is $A(x)$ times a factor which becomes small with large T . From (24), we also have that for some constant C and

all sufficiently large x ,

$$\sum_{k: A_k \subset J_{\delta'}} u_k^{-1} e^{-u_k^2/2} \leq CT^{-1}A(x). \quad (29)$$

Denote by $p_{k,l}$, the term of the double sum in (26) with index k and l . Similarly to (19), with $b = (1 - \varepsilon)B$, we have

$$\begin{aligned} p_{k,l} &= P\left(\max_{t \in A'_k} Z_x(t) > \frac{1}{\sqrt{r(x)}}, \max_{t \in A'_l} Z_x(t) > \frac{1}{\sqrt{r(x)}}\right) \\ &\leq P\left(\max_{t \in A'_k} \frac{Z_x(t)}{\sigma(t,x)} > u_k, \max_{t \in A'_l} \frac{Z_x(t)}{\sigma(t,x)} > u_l\right) \\ &\leq P\left(\max_{t \in A'_k} \frac{Z_x(t)}{\sigma(t,x)} > u_{k,l}, \max_{t \in A'_l} \frac{Z_x(t)}{\sigma(t,x)} > u_{k,l}\right) \\ &= P\left(\max_{t \in A'_0} Y_x(t) > u_{k,l}, \max_{t \in A'_{l-k}} Y_x(t) > u_{k,l}\right), \end{aligned}$$

where $u_{k,l} = \min\{u_k, u_l\}$. By the first statement of Lemma 4 we have, for any $T, \kappa > 0$ that there exists a constant $c > 0$ such that for all t, s with $T \geq |t - s| \geq \kappa$,

$$\frac{1}{r(x)}(1 - \rho(g(x)t, g(x)s)) \geq c|t - s|^{2-a}, \quad (30)$$

where ρ is the covariance function of Y_x . In addition, by the second statement of Lemma 4 we have that there exists a constant $c_1 > 0$ such that for all t, s with $\delta/g(x) \geq |t - s| \geq 1$,

$$\frac{1}{r(x)}(1 - \rho(g(x)t, g(x)s)) \geq c_1|t - s|^{2-a'}. \quad (31)$$

Let us estimate the sum of $p_{k,l}$ on $|k - l| = 1$ corresponding to neighboring intervals. By Lemma 9 and (30)

$$\begin{aligned} p_{k,k+1} &\leq \sum_{t \in A'_0, s \in A'_1} P(Y_x(t) > u_{k,k+1}, Y_x(s) > u_{k,k+1}) \\ &\leq \Psi(u_{k,k+1}) \sum_{i=1}^{\infty} i e^{-1/4c(\kappa i)^{2-a}} \leq \frac{C_1}{\kappa^2} \Psi(u_{k,k+1}), \end{aligned}$$

for some generic $C > 0$. So by (29)

$$\sum_{|k-l|=1} p_{k,l} \leq \frac{C}{\kappa^2 T} A(x). \quad (32)$$

Now consider the sum of all $p_{k,l}$ with $|k - l| = m > 1$. In order to use (31) suppose that $T > 1$. By Lemmas 4 and 9 we have, uniformly in all $m > 1$ such

that $mTg(x) \leq \delta'$,

$$\begin{aligned} p_{k,k+m} &\leq \frac{T^2}{\kappa^2} \max_{t \in \Delta'_0, s \in \Delta'_m} \mathbb{P}(Y_x(t) > u_{k,k+m}, Y_x(s) > u_{k,k+m}) \\ &\leq \frac{CT^2}{\kappa^2} e^{-c_1((m-1)T)^{2-a'}/4} \Psi(u_{k,k+m}) \end{aligned}$$

because there are T/κ many points in any of the intervals Δ'_k, Δ'_{k+m} which are separated by at least $(m-1)Tg(x)$. Therefore, for some C and all sufficiently large x , by (29) again

$$\sum_{|k-l|=m} p_{k,l} \leq \frac{CT}{\kappa^2} e^{-c_1((m-1)T)^{2-a'}/8} A(x) \leq \frac{C}{\kappa^2} e^{-c_1((m-1)T)^{2-a'}/8} A(x). \quad (33)$$

Now from (32) and (33), summing over m , we get that the double sum is not greater than $CT^{-1}\kappa^{-2}A(x)$, for some C and all sufficiently large x . Dividing all terms of (26) by $A(x)$, taking then \liminf as $x \rightarrow \infty$ and finally letting $T \rightarrow \infty$, we get asymptotically a lower bound for our probability. It can be made arbitrarily close to the upper bound by choosing sufficiently small κ and δ . Thus we have that for a small $\delta > 0$,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{A(x)} \mathbb{P} \left(\max_{t \in [t_0 - \delta, t_0 + \delta]} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right) \\ = \frac{\sqrt{\pi} H_{2-a} \sigma}{\sqrt{B((1-a)(2-a)\sigma^4/a^2)^{1/(2-a)}}}. \end{aligned} \quad (34)$$

(D) *Remaining approximation:* To conclude the proof of Theorem 1 choose a small positive ϵ and δ . Then choose T ($> \max(1, t_0)$) as large as $Dr([xT]) < 1.8\sigma_x^2 r(x)$ for all sufficiently large x . This is possible since $r(x)$ is regularly varying. For such T we get by Lemma 5

$$\mathbb{P} \left(\max_{t \geq T} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right) = O \left(\exp \left(-\frac{1}{1.8\sigma_x^2 r(x)} \right) \right) = o \left(\Psi \left(\frac{1}{\sqrt{r(x)}\sigma_x} \right) \right)$$

as $x \rightarrow \infty$. This is of smaller order than the asymptotic approximation of $\mathcal{P}(x)$.

Left is to consider the probability corresponding to the remaining interval

$$\mathbb{P} \left(\sup_{t \in [0, T] \setminus [t_0 - \delta', t_0 + \delta']} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right).$$

By (18), for some $\delta'' > 0$ and all sufficiently large x ,

$$\sup_{t \in [0, T] \setminus [t_0 - \delta', t_0 + \delta']} \text{Var } Y_x(t) = \sup_{t \in [0, T] \setminus [t_0 - \delta, t_0 + \delta]} \frac{\sigma^2(t, x)}{\sigma_x^2} \leq 1 - \delta''$$

therefore using the property of the third statement of Lemma 4 for Z_x and hence also for Y_x with Fernique's inequality, we get

$$\mathbb{P} \left(\sup_{t \in [0, T] \setminus [t_0 - \delta', t_0 + \delta']} Z_x(t) > \frac{1}{\sqrt{r(x)}} \right) = O \left(\exp \left(-\frac{1 + \vartheta}{\sigma_x^2 r(x)} \right) \right),$$

for some positive ϑ .

Thus, these remaining approximations are asymptotically of lower order than the expression (34) which shows our claim.

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